Some Measures on the Standard Bivariate Lognormal Distribution

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Abstract: While univariate and bivariate lognormal distributions have demonstrated great utility in a number of applications related to decision sciences, practitioners find few – if any – tables of their cumulative distributions function available to support their work. This paper describes a "standardized" form of the univariate and bivariate lognormal distributions and a methodology by which tables of their cumulative distribution functions can be generated. This paper, then, provides these reference tables and illustrates their use. Finally, it is noted that this methodology may be readily extended to the multivariate lognormal distribution.

INTRODUCTION

A lognormally-distributed random variable is a random variable whose logarithm is normally-distributed. Johnson, *et. al.*, (1994) note that some practitioners maintain "that the lognormal distribution is as fundamental as the normal distribution" and that the lognormal distribution has found applications in fields including the physical sciences, life sciences, social sciences, and engineering. Aitchison and Brown (1957) is the classic reference on the lognormal distribution, and the compendium edited by Crow and Shimizu (1988) provides a more recent update. Thomopoulos and Johnson (2003) present a full set of reference tables on the standard lognormal distribution and illustrate their use.

Despite the lognormal distribution's utility, practitioners find few – if any – tables of its cumulative distribution function available to support their work. Moshman (1953) has published selected upper and lower percentile points (0.5%, 1%, 2.5%, 5%, and 10%) as a function of the shape parameter. Similarly, Broadbent (1956) provides upper and lower 1% and 5% values as a function of the coefficient of variation. Thomopoulos and Longinow (1984) utilize the bivariate lognormal distribution in a structural reliability application. It is the objective of this paper to provide practitioners with more comprehensive tables of the cumulative distribution function of the univariate and bivariate lognormal distributions. Additionally, we illustrate a methodology by which "critical values" of "standardized" lognormal distributions may be determined.

CHARACTERISTICS OF LOGNORMAL DISTRIBUTIONS

Univariate Lognormal Distribution

Consider a lognormally-distributed variable x with mean μ_x and standard deviation σ_x – denoted $LN(\mu_x, \sigma_x^2)$. The variable y, where

$$y = \ln(x), \tag{1}$$

is normally-distributed with mean μ_y and standard deviation σ_y and is denoted $N(\mu_y, \sigma_y^2)$. The probability density function *f(x)* of the lognormal distribution is given by

$$f(x) = \frac{1}{x\sigma_y\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{\ln x - \mu_y}{\sigma_y}\right]^2} \quad (2)$$

as noted, for example, in Hines and Montgomery (1990). This distribution is skewed with a longer tail to the right of the mean.

Aitchison and Brown (1957) note that, when μ_y and σ_y are known for *y*, the corresponding mean and variance for *x* can be found from the following:

$$\mu_x = e^{\mu_y + \frac{1}{2}\sigma_y^2} \quad (3)$$

$$\sigma_x^2 = e^{2\mu_y + \sigma_y^2}\left(e^{\sigma_y^2} - 1\right) \quad (4)$$

Similarly, using Equation (1), when μ_x and σ_x are known for *x*, the corresponding mean and variance for *y* can be determined from the following:

$$\mu_y = \ln\left(\frac{\mu_x^2}{\sqrt{\mu_x^2 + \sigma_x^2}}\right) \quad (5)$$

$$\sigma_y^2 = \ln\left(1 + \frac{\sigma_x^2}{\mu_x^2}\right) \quad (6)$$

Johnson, *et. al.*, (1994) note various ways in which the lognormal distribution has been standardized. In this paper, we study the properties of the "standardized lognormal distribution" that arises when the mean of its normal counterpart is zero – i.e., $\mu_y = 0$ so that *y* is $N(0, \sigma_y^2)$. For this case, the mean and variance of *x* become

$$\mu_x = e^{\frac{1}{2}\sigma_y^2} \quad (7)$$

$$\sigma_x^2 = e^{\sigma_y^2}\left(e^{\sigma_y^2} - 1\right) \quad (8)$$

In the event that the mean of *y* is not equal to zero, the random variable can be transformed into standardized form *y'* as follows:

$$y' = y - \mu_y \quad (9)$$

We use this standardized lognormal distribution form to develop tables for the cumulative distribution function *F(x)* in the next section.

Bivariate Lognormal Distribution

Consider the variables x_1 and x_2 and $y_1 = \ln(x_1)$ and $y_2 = \ln(x_2)$ with x_1 and x_2 bivariate normal. The mean and standard deviation of y_1 and y_2 are μ_{y1}, σ_{y1}, μ_{y2} and σ_{y2}, respectively. The correlation coefficient between y_1 and y_2 is ρ_y. The pair x_1 and x_2 are distributed as bivariate lognormal and have the distribution LN(μ_{y1}, μ_{y2}, σ_{y1}, σ_{y2}, ρ_y).

When $\mu_{y1}=0$ and $\mu_{y2}=0$, x_1 and x_2 are "standardized" lognormal variables, and the distribution is LN(0, 0, σ_{y1}, σ_{y2}, ρ_y). We use this standardized lognormal distribution form to develop tables for the cumulative distribution function $F(x_1, x_2)$ in the next section.

Law and Kelton (2000) show the covariance and correlation of the bivariate lognormal variables x_1 and x_2 as follows:

$$\sigma_{x_1 x_2} = e^{\sigma_{y1}\sigma_{y2} - 1} e^{\mu_{y1} + \mu_{y2} + \frac{\sigma_{y1}^2 + \sigma_{y2}^2}{2}} \tag{10}$$

$$\rho_x = \frac{e^{\sigma_{y1}\sigma_{y2}} - 1}{\sqrt{e^{\sigma_{y1}^2 - 1} e^{\sigma_{y2}^2 - 1}}} \tag{11}$$

They further show that the covariance for the related pair of variables y_1 and y_2 from the bivariate normal distribution is given by

$$\sigma_{y1y2} = \ln\left(1 + \frac{\sigma_{x1x2}}{|\mu_{x1}\mu_{x2}|}\right) \tag{12}$$

TABLES OF THE CUMULATIVE DISTRIBUTION FUNCTIONS

Univariate Lognormal Distribution

Table 1 shows the shape of a univariate lognormal variable x when y is normal with $\mu_y = 0$ and for σ_y values of 0.5(0.5)2.5. For comparison sake, the values of x are listed with associated values of z from the standard normal distribution for z values of -3.0(0.5)3.0 – along with the associated cumulative distribution values of *F(z)*. Note that the smaller values of σ_y correspond with a shape of the lognormal x that is the least skewed – whereas, when $\sigma_y = 2.5$, the variable x is extremely skewed to the right.

Bivariate Lognormal Distribution

Table 2 lists the relation between the correlation of the bivariate normal pair (y_1, y_2) with the bivariate lognormal pair (x_1, x_2). The correlation of (y_1, y_2) is ρ_y, and the correlation of (x_1, x_2) is ρ_x.

Given particular values for a standard bivariate normal pair (z_1, z_2), a unique set of corresponding variables can be obtained from the transformations

$$y_1 = \sigma_{y1} z_1 \text{ and } y_2 = \sigma_{y2} z_2 \qquad (13)$$

and

$$x_1 = e^{y1} \text{ and } x_2 = e^{y2} \qquad (14)$$

Therefore, the associated set of values (z_1, z_2), (y_1, y_2), and (x_1, x_2) have identical cumulative probabilities. That is,

$$F(z_1, z_2) = F(y_1, y_2) = F(x_1, x_2) \qquad (15)$$

In Equation 15, the cumulative distribution functions are:

$F(z_1, z_2)$ for the standard bivariate normal variables (z_1, z_2) with distribution N(0, 0, 1, 1, ρ_z).

$F(y_1, y_2)$ for the standard bivariate normal variables (y_1, y_2) with distribution N(0, 0, σ_{y1}, σ_{y2}, ρ_y).

$F(x_1, x_2)$ for the standard bivariate lognormal variables (x_1, x_2) with distribution LN(0, 0, σ_{y1}, σ_{y2}, ρ_y).

Table 3 shows the values of $F(z_1, z_2)$ for values of ρ_z = 0.0(0.3)0.9 and for all combinations of (z_1, z_2) with values of -3.0(0.5)3.0. A more detailed listing of reference tables on the standard bivariate normal distribution are presented in Jantaravareerat, M. and N. Thomopoulos (1998). Note that ρ_y is always the same as ρ_z; however, ρ_x would be found from Table 2. The corresponding pairs (z_1, z_2), (y_1, y_2), and (x_1, x_2) are related by Equations 13 and 14.

Multivariate Lognormal Distribution.

Lindee and Thomopoulos (2001) have developed reference tables for the cumulative distribution function of the standard multivariate normal distribution. It is noted that reference tables for the standard multivariate lognormal distribution can be developed in a similar way as shown in this paper for the standard bivariate lognormal distribution.

Use of the Tables

Suppose (x_1, x_2) are bivariate lognormal with distribution LN(0, 0, 1.5, 2.0, 0.8) with x_1 = 20 and x_2 = 7. To find the approximate cumulative probability of F(20, 7), we note – from Table 1 – that z_1 ~ 2.0 and z_2 ~ 1.0. Table 3, then, gives F(2.0, 1.0) = 0.84 for ρ = 0.8. Thus, for the lognormal pair F(20, 7) ~ 0.84.

Similarly, to find the value of x_2 where $F(20, x_2) \sim 0.95$ for this distribution, we note – from Table 3 – that $F(2.0, z_2) = 0.95$ when $z_2 \sim 1.8$. Therefore, $y_2 = 2.0\times1.8 = 3.6$ and $x_2 = e^{3.6} = 36.6$. So, for the lognormal pair (x_1, x_2), $F(20, 36.6) \sim 0.95$.

CONCLUSION

This paper gives various measures on the univariate and bivariate lognormal distributions. We show how the lognormal variable is related to a normal variable and its associated standard normal variable. In the same way, the standard bivariate lognormal variables are related to a pair of bivariate normal variables and a pair of standard bivariate variables. Tables listing representative lognormal variables are given, and tables relating the correlation between the lognormal variables and normal variables are presented. Finally, cumulative distribution tables are listed for the standard bivariate normal and standard bivariate lognormal distributions. We also note how this methodology may be extended to the multivariate lognormal distribution.

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Table 1. Five representations of the standard lognormal variable (x) as related to the normal variable y with mean zero and standard deviation σ_y and to the standard normal variable z and its cumulative distribution $F(z)$.

				σ_y		
z	**F(z)**	**0.5**	**1**	**1.5**	**2**	**2.5**
-3.0	**0.001**	0.22	0.05	0.01	0.00	0.00
-2.5	**0.006**	0.29	0.08	0.02	0.01	0.00
-2.0	**0.023**	0.37	0.14	0.05	0.02	0.01
-1.5	**0.067**	0.47	0.22	0.11	0.05	0.02
-1.0	**0.159**	0.61	0.37	0.22	0.14	0.08
-0.5	**0.309**	0.78	0.61	0.47	0.37	0.29
0.0	**0.500**	1.00	1.00	1.00	1.00	1.00
0.5	**0.691**	1.28	1.65	2.12	2.72	3.49
1.0	**0.841**	1.65	2.72	4.48	7.39	12.18
1.5	**0.933**	2.12	4.48	9.49	20.09	42.52
2.0	**0.977**	2.72	7.39	20.09	54.60	148.41
2.5	**0.994**	3.49	12.18	42.52	148.41	518.01
3.0	**0.999**	4.48	20.09	90.02	403.43	1808.04

Table 2. The correlation ρ_x of (x_1, x_2) from the standard bivariate lognormal distribution as it is related to the correlation ρ_y of (y_1, y_2) of the bivariate normal distribution and standard deviations σ_{y1} and σ_{y2}.

							ρ_y					
σ_{y1}	σ_{y2}	**0.0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**
0.5	**0.5**	0.00	0.09	0.18	0.27	0.37	0.47	0.57	0.67	0.78	0.89	1.00
0.5	**1.0**	0.00	0.07	0.15	0.23	0.32	0.41	0.50	0.60	0.70	0.81	0.93
0.5	**1.5**	0.00	0.05	0.10	0.16	0.23	0.29	0.37	0.44	0.53	0.62	0.72
0.5	**2.0**	0.00	0.03	0.06	0.09	0.13	0.17	0.21	0.26	0.31	0.37	0.44
0.5	**2.5**	0.00	0.01	0.02	0.04	0.05	0.07	0.09	0.12	0.14	0.17	0.21
1.0	**0.5**	0.00	0.07	0.15	0.23	0.32	0.41	0.50	0.60	0.70	0.81	0.93
1.0	**1.0**	0.00	0.06	0.13	0.20	0.29	0.38	0.48	0.59	0.71	0.85	1.00
1.0	**1.5**	0.00	0.04	0.09	0.15	0.22	0.29	0.38	0.49	0.61	0.75	0.91
1.0	**2.0**	0.00	0.02	0.05	0.09	0.13	0.18	0.24	0.32	0.41	0.53	0.67
1.0	**2.5**	0.00	0.01	0.02	0.04	0.06	0.08	0.12	0.16	0.21	0.28	0.38
1.5	**0.5**	0.00	0.05	0.10	0.16	0.23	0.29	0.37	0.44	0.53	0.62	0.72
1.5	**1.0**	0.00	0.04	0.09	0.15	0.22	0.29	0.38	0.49	0.61	0.75	0.91
1.5	**1.5**	0.00	0.03	0.07	0.11	0.17	0.25	0.34	0.45	0.59	0.77	1.00
1.5	**2.0**	0.00	0.02	0.04	0.07	0.11	0.16	0.24	0.34	0.47	0.65	0.89
1.5	**2.5**	0.00	0.01	0.02	0.03	0.05	0.08	0.13	0.19	0.29	0.43	0.63
2.0	**0.5**	0.00	0.03	0.06	0.09	0.13	0.17	0.21	0.26	0.31	0.37	0.44
2.0	**1.0**	0.00	0.02	0.05	0.09	0.13	0.18	0.24	0.32	0.41	0.53	0.67
2.0	**1.5**	0.00	0.02	0.04	0.07	0.11	0.16	0.24	0.34	0.47	0.65	0.89
2.0	**2.0**	0.00	0.01	0.02	0.04	0.07	0.12	0.19	0.29	0.44	0.66	1.00
2.0	**2.5**	0.00	0.00	0.01	0.02	0.04	0.07	0.11	0.19	0.32	0.53	0.89
2.5	**0.5**	0.00	0.01	0.02	0.04	0.05	0.07	0.09	0.12	0.14	0.17	0.21
2.5	**1.0**	0.00	0.01	0.02	0.04	0.06	0.08	0.12	0.16	0.21	0.28	0.38
2.5	**1.5**	0.00	0.01	0.02	0.03	0.05	0.08	0.13	0.19	0.29	0.43	0.63
2.5	**2.0**	0.00	0.00	0.01	0.02	0.04	0.07	0.11	0.19	0.32	0.53	0.89
2.5	**2.5**	0.00	0.00	0.00	0.01	0.02	0.04	0.08	0.15	0.29	0.53	1.00

Table 3. The cumulative distribution functions $F(z_1, z_2)$ of the standard bivariate normal distribution – and, thus, the related $F(y_1, y_2)$ from the bivariate normal distribution and $F(x_1, x_2)$ from the bivariate lognormal distribution for ρ_z values of 0(0.3)0.9.

ρ = 0.0

z_2/z_1	-3.0	-2.5	-2.0	-1.5	-1.0	-0.5	0.0	0.5	1.0	1.5	2.0	2.5	3.0
3.0	0.00	0.01	0.02	0.07	0.16	0.31	0.50	0.69	0.84	0.93	0.98	0.99	1.00
2.5	0.00	0.01	0.02	0.07	0.16	0.31	0.50	0.69	0.84	0.93	0.97	0.99	0.99
2.0	0.00	0.01	0.02	0.07	0.16	0.30	0.49	0.68	0.82	0.91	0.96	0.97	0.98
1.5	0.00	0.01	0.02	0.06	0.15	0.29	0.47	0.65	0.79	0.87	0.91	0.93	0.93
1.0	0.00	0.01	0.02	0.06	0.13	0.26	0.42	0.58	0.71	0.79	0.82	0.84	0.84
0.5	0.00	0.00	0.02	0.05	0.11	0.21	0.35	0.48	0.58	0.65	0.68	0.69	0.69
0.0	0.00	0.00	0.01	0.03	0.08	0.15	0.25	0.35	0.42	0.47	0.49	0.50	0.50
-0.5	0.00	0.00	0.01	0.02	0.05	0.10	0.15	0.21	0.26	0.29	0.30	0.31	0.31
-1.0	0.00	0.00	0.00	0.01	0.03	0.05	0.08	0.11	0.13	0.15	0.16	0.16	0.16
-1.5	0.00	0.00	0.00	0.00	0.01	0.02	0.03	0.05	0.06	0.06	0.07	0.07	0.07
-2.0	0.00	0.00	0.00	0.00	0.00	0.01	0.01	0.02	0.02	0.02	0.02	0.02	0.02
-2.5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.01	0.01	0.01	0.01
-3.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

ρ = 0.3

z_2/z_1	-3.0	-2.5	-2.0	-1.5	-1.0	-0.5	0.0	0.5	1.0	1.5	2.0	2.5	3.0
3.0	0.00	0.01	0.02	0.07	0.16	0.31	0.50	0.69	0.84	0.93	0.98	0.99	1.00
2.5	0.00	0.01	0.02	0.07	0.16	0.31	0.50	0.69	0.84	0.93	0.97	0.99	0.99
2.0	0.00	0.01	0.02	0.07	0.16	0.31	0.49	0.68	0.83	0.91	0.96	0.97	0.98
1.5	0.00	0.01	0.02	0.07	0.16	0.30	0.48	0.66	0.80	0.88	0.92	0.93	0.93
1.0	0.00	0.01	0.02	0.06	0.15	0.28	0.45	0.61	0.73	0.80	0.83	0.84	0.84
0.5	0.00	0.01	0.02	0.06	0.13	0.25	0.39	0.52	0.61	0.66	0.68	0.69	0.69
0.0	0.00	0.01	0.02	0.05	0.11	0.20	0.30	0.39	0.45	0.48	0.49	0.50	0.50
-0.5	0.00	0.00	0.01	0.04	0.08	0.14	0.20	0.25	0.28	0.30	0.31	0.31	0.31
-1.0	0.00	0.00	0.01	0.02	0.04	0.08	0.11	0.13	0.15	0.16	0.16	0.16	0.16
-1.5	0.00	0.00	0.00	0.01	0.02	0.04	0.05	0.06	0.06	0.07	0.07	0.07	0.07
-2.0	0.00	0.00	0.00	0.00	0.01	0.01	0.02	0.02	0.02	0.02	0.02	0.02	0.02
-2.5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.01	0.01	0.01	0.01	0.01
-3.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

ρ = 0.6

z_2/z_1	-3.0	-2.5	-2.0	-1.5	-1.0	-0.5	0.0	0.5	1.0	1.5	2.0	2.5	3.0
3.0	0.00	0.01	0.02	0.07	0.16	0.31	0.50	0.69	0.84	0.93	0.98	0.99	1.00
2.5	0.00	0.01	0.02	0.07	0.16	0.31	0.50	0.69	0.84	0.93	0.97	0.99	0.99
2.0	0.00	0.01	0.02	0.07	0.16	0.31	0.50	0.69	0.83	0.92	0.96	0.97	0.98
1.5	0.00	0.01	0.02	0.07	0.16	0.31	0.50	0.68	0.81	0.89	0.92	0.93	0.93
1.0	0.00	0.01	0.02	0.06	0.15	0.30	0.47	0.64	0.75	0.81	0.83	0.84	0.84
0.5	0.00	0.01	0.02	0.07	0.15	0.29	0.44	0.57	0.64	0.68	0.69	0.69	0.69
0.0	0.00	0.01	0.02	0.06	0.14	0.25	0.36	0.44	0.48	0.49	0.50	0.50	0.50
-0.5	0.00	0.01	0.02	0.05	0.11	0.18	0.24	0.28	0.30	0.31	0.31	0.31	0.31
-1.0	0.00	0.00	0.01	0.04	0.07	0.11	0.14	0.15	0.16	0.16	0.16	0.16	0.16
-1.5	0.00	0.00	0.01	0.02	0.04	0.05	0.06	0.07	0.07	0.07	0.07	0.07	0.07
-2.0	0.00	0.00	0.00	0.01	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
-2.5	0.00	0.00	0.00	0.00	0.00	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
-3.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

ρ = 0.9

z_2/z_1	-3.0	-2.5	-2.0	-1.5	-1.0	-0.5	0.0	0.5	1.0	1.5	2.0	2.5	3.0
3.0	0.00	0.00	0.02	0.07	0.16	0.31	0.50	0.69	0.84	0.93	0.98	0.99	1.00
2.5	0.00	0.00	0.02	0.07	0.16	0.31	0.50	0.69	0.84	0.93	0.98	0.99	0.99
2.0	0.00	0.00	0.02	0.07	0.16	0.31	0.50	0.69	0.84	0.93	0.97	0.98	0.98
1.5	0.00	0.01	0.02	0.07	0.16	0.31	0.50	0.70	0.84	0.91	0.93	0.93	0.93
1.0	0.00	0.00	0.02	0.06	0.16	0.30	0.49	0.67	0.79	0.83	0.84	0.84	0.84
0.5	0.00	0.01	0.02	0.07	0.16	3.10	0.50	0.63	0.68	0.69	0.69	0.69	0.69
0.0	0.00	0.01	0.02	0.07	0.16	0.31	0.44	0.49	0.50	0.50	0.50	0.50	0.50
-0.5	0.00	0.01	0.02	0.07	0.16	0.25	0.30	0.31	0.31	0.31	0.31	0.31	0.31
-1.0	0.00	0.00	0.02	0.06	0.11	0.15	0.16	0.16	0.16	0.16	0.16	0.16	0.16
-1.5	0.00	0.00	0.02	0.04	0.06	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
-2.0	0.00	0.00	0.01	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
-2.5	0.00	0.00	0.00	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
-3.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00