CHARACTERISTICS AND TABLES OF THE PARTIAL EXPECTATION OF THE LEFT-TRUNCATED NORMAL DISTRIBUTION

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ABSTRACT

Left-truncated normal distributions – i.e., normal probability distributions in which values below a truncation point cannot be observed – have found great utility in a variety of disciplines within the purview of decision science. This paper provides a relation between the truncation point and the left-truncated normal distribution's coefficient of variation. Through the introduction of a standardized truncated variable, a table of the partial expectation of the left-truncated normal distribution is developed and presented for reference.

INTRODUCTION

Left-truncated normal distributions – i.e., normal distributions in which values below a "truncation point" cannot be observed – have found great utility in a variety of disciplines within the purview of decision science. These fields include operations management (inventory planning, reliability analysis, and process management), regression analysis, and finance (capital budgeting, portfolio analysis, and options pricing). Johnson and Thomopoulos studied the characteristics of the left-truncated normal distribution and published reference tables of its cumulative distribution function.[2] They, then, utilized the partial expectation of the left-truncated normal distribution to develop an enhanced methodology for establishing safety stock levels in inventory management.[3] While Thomopoulos had published selected values of this partial expectation, the tabulation was limited and suffered from computational inaccuracies for high levels of truncation.[5] This paper extends and corrects this tabulation in order to provide a useful reference table of the partial expectation of the left-truncated normal distribution for practitioners. In the process of developing this table, this paper provides a relation between the coefficient of variation and the point of truncation.

REVIEW OF CHARACTERISTICS AND NOTATION

Consider a normally-distributed random variable x with a probability density function *f(x)* specified as

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty \le x \le \infty \tag{1}$$

If the values of x below some value x_L cannot be observed – due to censoring or truncation – then, following Hald's conventions, the resulting distribution is a left-truncated normal distribution with probability density function $f_{LTN}(x)$ given by

$$f_{LTN}(x) = \begin{cases} 0, & -\infty \le x \le x_L \\ \dfrac{f(x)}{\int_{x_L}^{\infty} f(x)dx}, & x_L \le x \le \infty \end{cases} \tag{2}$$

where *f(x)* is as defined in Equation 1.[1]

For purposes of generality, Equation 1 can be re-stated in terms of the standard normal distribution (denoted *f(z)*), where

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, -\infty \le z \le \infty \tag{3}$$

The point of truncation x_L can also be expressed in terms of the standard normal distribution and denoted by k_L.

Reformulating the left-truncated normal distribution of Equation 2 in terms of the standard normal distribution, the following can be found:

$$f_{LTN}(z) = \begin{cases} 0, & -\infty \le z \le k_L \\ \dfrac{f(z)}{H(k_L)}, & k_L \le z \le \infty \end{cases} \tag{4}$$

where $H(k_L)$ is introduced for simplicity and defined as follows:

$$H(k) = \int_{k}^{\infty} f(z)dz = 1 - F(k) \tag{5}$$

Similar expressions are found in Schneider, for example.[4]

To define what Thomopoulos terms as a "standardized, left-truncated normal distribution," a standardizing variable $t = z - k_L$ is introduced.[5] This has the effect of defining the point of truncation as $t = 0$. The standardized, left-truncated normal distribution $f_{SLTN}(t)$ is, thus, given by

$$f_{SLTN}(t) = \begin{cases} 0, & t \le 0 \\ \dfrac{f(t + k_L)}{H(k_L)}, & t \ge 0 \end{cases} \tag{6}$$

The mean μ_t of the standardized, left-truncated normal distribution is given by

$$\mu_t = \frac{1}{H(k_L)}\left[f(k_L) - k_L H(k_L)\right] \quad (7)$$

and the standard deviation σ_t of the standardized, left-truncated normal distribution is given by

$$\sigma_t^2 = E(t^2) - \mu_t^2 \quad (8)$$

where

$$E(t^2) = \frac{1}{H(k_L)}\left[\left(1 + k_L^2\right)H(k_L) - k_L f(k_L)\right] \quad (9)$$

COEFFICIENT OF VARIATION

Both the mean and standard deviation – μ_t and σ_t in Equations 7 and 8, respectively – are uniquely determined by and solely dependent upon the truncation point k_L.[2] Consequently, the coefficient of variation c of the standardized, left-truncated normal distribution exists uniquely for a particular k_L. This relationship is illustrated in Figure 1.

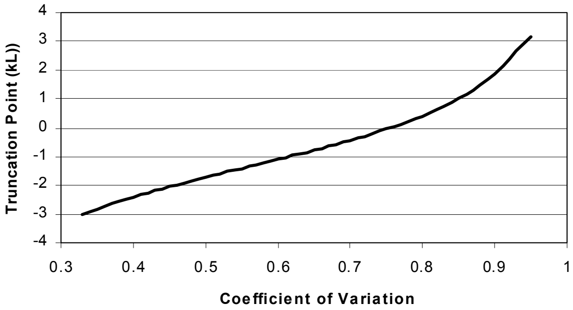

Figure 1 – Truncation point (k_L) as a function of the coefficient of variation.

In practice, a number of observed random variables are truncated at zero – e.g., inventory demands, processing times, etc. (Where a pattern of variation is truncated at a value other than zero, this value may be subtracted from the mean prior to the calculation of the coefficient of variation.) Given this, particularly when the sample sizes are large, the sample mean and standard deviation may be used to calculate an approximate value of the coefficient of variation c. Figure 1 may, then, be used to estimate the truncation point k_L. Alternatively, the truncation point may be approximated by the following function

$$k_L = 77.417c^4 - 159.03c^3 + 117.3c^2 - 29.875c - 1.0641 \quad (10)$$

The standard deviation of the errors associated with this approximation over the range $-3 \le k_L \le 3$ is 0.031 – which is adequate for most purposes. (A fifth-order polynomial approximation reduces the standard deviation of the errors to 0.014.)

PARTIAL EXPECTATION

For the standard normal variate (z), the expected value of the standard normal variate beyond the value z_0 is commonly called the "partial expectation of z beyond z_0" and is denoted $E(z > z_0)$. This is given by

$$E(z > z_0) = \int_{z_0}^{\infty} (z - z_0) f(z)\,dz = f(z_0) - z_0 H(z_0) \quad (11)$$

As noted earlier, the truncation point k_L uniquely determines the standardized left-truncated normal distribution's mean μ_t and standard deviation σ_t (i.e., its coefficient of variation).

Define a standardized truncated variable w as follows

$$w = \frac{t - \mu_t}{\sigma_t} \quad (12)$$

Calculation of $E(w > w_0)$ in terms of $E(t > t_0)$ follows directly from Equation 12 with

$$E(w > w_0) = \frac{E(t > t_0)}{\sigma_t} \quad (13)$$

Similarly, $E(t > t_0)$ – for a particular truncation point k_L – can be determined using

$$E(t > t_0)\Big|_{k_L} = \frac{E(z > z_0)}{H(k_L)} \quad (14)$$

Combining Equations 13 and 14 results in

$$E(w > w_0) = \frac{E(z > z_0)}{H(k_L)\sigma_t} \quad (15)$$

with

$$z_0 = \mu_t + w_0\sigma_t + k_L \quad (16)$$

Thus, for a particular (left-truncated) pattern of variability – defined by the sample mean and standard deviation – the partial expectation is given by Equation 16.

Development of the Table

This paper now presents a table for determining the partial expectation of w beyond w_0 – i.e., $E(w > w_0)$. Table 1 lists the value of $E(w > w_0)$ associated with a particular value of w_0 as a function of the coefficient of variation c. For reference, the truncation point k_L associated with c is also given. Values of $E(w > w_0)$ have been suppressed for $w_0 < w_{min}$ – where w_{min} is given by

$$w_{\min} = -\frac{\mu_t}{\sigma_t} \quad (17)$$

Use of the Table

To illustrate the use of Table 1, consider an inventory manager for a product whose average lead-time demand is 50 units with a standard deviation of 40 units. In establishing an acceptable level of safety stock, she needs to determine the expected number of pieces short for a given stock level – say, of 90 units.

From this problem statement, the coefficient of variation *c* can be determined to be 0.8. In terms of the standardized truncated variable *w*, the stock level of 90 units is equivalent to $w_0 = 1$. From the *c* = 0.8 column in Table 1, we find $E(w > 1)$ to be 0.1256.

CONCLUSION

This paper provided a relation between the truncation point and the left-truncated normal distribution's coefficient of variation. Through the introduction of a standardized truncated variable, a table of the partial expectation of the left-truncated normal distribution was developed and presented for reference.

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c	0.35	0.4	0.45	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95
w_0/k_L	-2.82	-2.41	-2.05	-1.73	-1.42	-1.10	-0.78	-0.43	-0.05	0.41	1.00	1.87	3.15
-3.0	***	***	***	***	***	***	***	***	***	***	***	***	***
-2.9	***	***	***	***	***	***	***	***	***	***	***	***	***
-2.8	2.8000	***	***	***	***	***	***	***	***	***	***	***	***
-2.7	2.7001	***	***	***	***	***	***	***	***	***	***	***	***
-2.6	2.6003	***	***	***	***	***	***	***	***	***	***	***	***
-2.5	2.5007	***	***	***	***	***	***	***	***	***	***	***	***
-2.4	2.4012	2.4001	***	***	***	***	***	***	***	***	***	***	***
-2.3	2.3020	2.3005	***	***	***	***	***	***	***	***	***	***	***
-2.2	2.2031	2.2012	2.2000	***	***	***	***	***	***	***	***	***	***
-2.1	2.1046	2.1024	2.1004	***	***	***	***	***	***	***	***	***	***
-2.0	2.0065	2.0040	2.0013	2.0000	***	***	***	***	***	***	***	***	***
-1.9	1.9090	1.9063	1.9030	1.9004	***	***	***	***	***	***	***	***	***
-1.8	1.8122	1.8093	1.8055	1.8019	1.8000	***	***	***	***	***	***	***	***
-1.7	1.7162	1.7132	1.7090	1.7045	1.7010	***	***	***	***	***	***	***	***
-1.6	1.6212	1.6181	1.6136	1.6084	1.6036	1.6005	***	***	***	***	***	***	***
-1.5	1.5273	1.5242	1.5195	1.5138	1.5079	1.5030	1.5002	***	***	***	***	***	***
-1.4	1.4348	1.4317	1.4270	1.4209	1.4142	1.4078	1.4028	1.4002	***	***	***	***	***
-1.3	1.3438	1.3408	1.3361	1.3299	1.3227	1.3152	1.3084	1.3031	1.3003	***	***	***	***
-1.2	1.2545	1.2518	1.2472	1.2410	1.2335	1.2253	1.2172	1.2099	1.2042	1.2007	***	***	***
-1.1	1.1673	1.1647	1.1604	1.1543	1.1468	1.1383	1.1294	1.1207	1.1128	1.1064	1.1020	1.1001	***
-1.0	1.0822	1.0799	1.0760	1.0702	1.0629	1.0543	1.0450	1.0354	1.0261	1.0176	1.0103	1.0048	1.0012
-0.9	0.9995	0.9976	0.9941	0.9888	0.9819	0.9735	0.9642	0.9542	0.9440	0.9342	0.9250	0.9168	0.9099
-0.8	0.9196	0.9181	0.9150	0.9104	0.9040	0.8961	0.8870	0.8770	0.8666	0.8560	0.8456	0.8357	0.8265
-0.7	0.8426	0.8414	0.8390	0.8350	0.8293	0.8221	0.8136	0.8040	0.7936	0.7828	0.7719	0.7611	0.7502
-0.6	0.7686	0.7679	0.7661	0.7629	0.7580	0.7517	0.7439	0.7350	0.7251	0.7145	0.7035	0.6924	0.6806
-0.5	0.6981	0.6978	0.6966	0.6942	0.6903	0.6849	0.6781	0.6701	0.6609	0.6509	0.6403	0.6292	0.6170
-0.4	0.6310	0.6311	0.6306	0.6290	0.6261	0.6218	0.6161	0.6091	0.6010	0.5918	0.5818	0.5713	0.5591
-0.3	0.5676	0.5681	0.5683	0.5675	0.5657	0.5625	0.5580	0.5522	0.5451	0.5370	0.5279	0.5181	0.5063
-0.2	0.5080	0.5089	0.5097	0.5098	0.5089	0.5069	0.5036	0.4990	0.4932	0.4863	0.4783	0.4694	0.4582
-0.1	0.4522	0.4535	0.4549	0.4558	0.4559	0.4550	0.4530	0.4497	0.4452	0.4395	0.4327	0.4249	0.4144
0.0	0.4004	0.4021	0.4039	0.4056	0.4067	0.4069	0.4060	0.4040	0.4007	0.3963	0.3908	0.3842	0.3746
0.1	0.3526	0.3545	0.3568	0.3592	0.3611	0.3623	0.3626	0.3618	0.3598	0.3567	0.3524	0.3470	0.3383
0.2	0.3087	0.3108	0.3136	0.3165	0.3192	0.3213	0.3226	0.3230	0.3222	0.3204	0.3173	0.3131	0.3054
0.3	0.2686	0.2710	0.2740	0.2774	0.2808	0.2838	0.2860	0.2874	0.2878	0.2871	0.2853	0.2823	0.2756
0.4	0.2324	0.2348	0.2381	0.2419	0.2458	0.2495	0.2526	0.2550	0.2564	0.2568	0.2561	0.2542	0.2485
0.5	0.1997	0.2023	0.2057	0.2099	0.2142	0.2184	0.2223	0.2254	0.2278	0.2292	0.2295	0.2286	0.2239
0.6	0.1706	0.1732	0.1767	0.1810	0.1857	0.1904	0.1948	0.1987	0.2018	0.2041	0.2053	0.2055	0.2016
0.7	0.1448	0.1473	0.1509	0.1553	0.1602	0.1652	0.1700	0.1745	0.1783	0.1813	0.1834	0.1844	0.1815
0.8	0.1220	0.1245	0.1280	0.1325	0.1374	0.1426	0.1478	0.1527	0.1571	0.1608	0.1636	0.1654	0.1632
0.9	0.1022	0.1045	0.1080	0.1123	0.1173	0.1226	0.1280	0.1332	0.1380	0.1422	0.1457	0.1481	0.1467
1.0	0.0849	0.0872	0.0905	0.0947	0.0996	0.1049	0.1104	0.1158	0.1209	0.1256	0.1295	0.1326	0.1318
1.1	0.0701	0.0722	0.0753	0.0794	0.0841	0.0893	0.0948	0.1003	0.1056	0.1106	0.1149	0.1185	0.1183
1.2	0.0575	0.0594	0.0623	0.0661	0.0706	0.0757	0.0811	0.0866	0.0920	0.0972	0.1018	0.1058	0.1062
1.3	0.0468	0.0485	0.0512	0.0547	0.0590	0.0638	0.0690	0.0745	0.0799	0.0852	0.0901	0.0944	0.0952
1.4	0.0378	0.0394	0.0418	0.0450	0.0490	0.0535	0.0585	0.0638	0.0692	0.0745	0.0796	0.0841	0.0853
1.5	0.0303	0.0317	0.0339	0.0368	0.0404	0.0447	0.0494	0.0545	0.0598	0.0651	0.0702	0.0749	0.0764
1.6	0.0241	0.0253	0.0273	0.0299	0.0332	0.0371	0.0415	0.0464	0.0515	0.0567	0.0618	0.0666	0.0684
1.7	0.0190	0.0201	0.0218	0.0241	0.0271	0.0306	0.0347	0.0393	0.0442	0.0492	0.0543	0.0591	0.0612
1.8	0.0149	0.0158	0.0173	0.0193	0.0220	0.0252	0.0289	0.0332	0.0378	0.0427	0.0476	0.0525	0.0547
1.9	0.0116	0.0124	0.0136	0.0154	0.0177	0.0206	0.0240	0.0279	0.0323	0.0369	0.0417	0.0465	0.0488
2.0	0.0089	0.0096	0.0107	0.0122	0.0142	0.0167	0.0198	0.0234	0.0274	0.0318	0.0365	0.0412	0.0436
2.1	0.0068	0.0074	0.0083	0.0096	0.0113	0.0135	0.0163	0.0195	0.0233	0.0274	0.0318	0.0364	0.0389
2.2	0.0052	0.0056	0.0064	0.0075	0.0090	0.0109	0.0133	0.0162	0.0197	0.0235	0.0277	0.0322	0.0347
2.3	0.0039	0.0043	0.0049	0.0058	0.0070	0.0087	0.0108	0.0135	0.0166	0.0202	0.0241	0.0284	0.0309
2.4	0.0029	0.0032	0.0037	0.0045	0.0055	0.0069	0.0088	0.0111	0.0139	0.0172	0.0210	0.0250	0.0275
2.5	0.0022	0.0024	0.0028	0.0034	0.0043	0.0055	0.0071	0.0091	0.0117	0.0147	0.0182	0.0220	0.0245
2.6	0.0016	0.0018	0.0021	0.0026	0.0033	0.0043	0.0057	0.0075	0.0097	0.0125	0.0157	0.0194	0.0218
2.7	0.0012	0.0013	0.0016	0.0019	0.0025	0.0034	0.0045	0.0061	0.0081	0.0106	0.0136	0.0170	0.0194
2.8	0.0008	0.0009	0.0011	0.0015	0.0019	0.0026	0.0036	0.0050	0.0067	0.0090	0.0117	0.0149	0.0172
2.9	0.0006	0.0007	0.0008	0.0011	0.0015	0.0020	0.0029	0.0040	0.0056	0.0076	0.0101	0.0131	0.0153
3.0	0.0004	0.0005	0.0006	0.0008	0.0011	0.0016	0.0023	0.0032	0.0046	0.0064	0.0087	0.0115	0.0135

Table 1 – Partial expectation $E(w > w_0)$ as a function of the coefficient of variation.