VALUES FOR THE CUMULATIVE DISTRIBUTION FUNCTION OF THE STANDARD MULTIVARIATE NORMAL DISTRIBUTION

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ABSTRACT

This paper provides cumulative distribution function values for the standard multivariate normal distribution. The values are derived from a simulation model for the multivariate normal distribution, which can be run on any standard personal computer. Utilizing user-input distribution parameters, the model simulates the actual standard multivariate normal distribution. Since approximations of the standard multivariate normal distribution are not used, the results are very accurate. The cumulative distribution function values can be found using a newly created Microsoft Excel function that allows exceptional flexibility for the user. The cumulative distribution function values can also be located on tables that were developed utilizing the simulation methods.

INTRODUCTION

In analysis of distributions that are normally distributed, the density function and the cumulative distribution function are often utilized to determine the probability of a certain value of the function, or a cumulative probability through a certain value of the function, respectively. The distribution is often converted to the standard normal distribution in order to use standardized tables that are found in just about every statistics book that is currently published.

The normal distribution provides explanation for the behavior of one variable that is normally distributed. The researcher can utilize the standardized cumulative distribution function tables found in most statistics references. Often times, however, multiple variables are present in a study. These variables can each possess the characteristics of a normal distribution individually, and are related to each other by correlation coefficients. A researcher could analyze each of these variables individually using the standard normal distribution tables, but there is no straightforward means to study the interaction of these variables and how they behave as a group. This behavior is the multivariate normal distribution. This paper provides, in very straightforward ways, the means by which a researcher can obtain the values for the cumulative standard multivariate normal distribution, for use in their analysis of multivariate normal distributions.

The general notation used in this paper is described here. Within the multivariate normal distribution there are k normal distributions (or k dimensions). The multivariate normal distribution has two parameters, mean μ and standard deviation σ, for each dimension, plus k(k-1)/2 correlation coefficients. For example, for k = 3, the parameters would be μ_1, μ_2, μ_3, σ_1, σ_2, σ_3, ρ_{12}, ρ_{13}, and ρ_{23}.

The values for the cumulative distribution function for the standard multivariate normal distribution are derived using two processes. The simulation methodology provided in this paper was the process used to generate the majority of the standard multivariate normal cumulative distribution function values. For an "endpoint" value, however, a different methodology was used. These methodologies will both be described in this paper. The paper also gives information on how to convert the observed multivariate normal distribution into the standard multivariate normal distribution, so that the tables and Excel functions can be used. Finally, the limitations for the use of these tools will be presented.

STATISTICAL BACKGROUND THEORY

For the multivariate normal distribution, the density function can be written as

$$f(x_1, x_2, \ldots, x_k) = \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} e^{-(1/2)(X-\mu)^T \Sigma^{-1}(X-\mu)} \quad (1)$$

where T denotes the transpose of the matrix, the –1 denotes the inverse of the matrix, Σ denotes the variance-covariance matrix, $|\Sigma|$ denotes the determinant of the variance-covariance matrix, **X** is the vector of x_k values, and μ is the vector of the means of the k distributions [1].

Covariance for two variables X_i and X_j is defined as $Cov(X_i, X_j) = \sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$. Subsequently, the variance-covariance matrix is defined as

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1k}\sigma_1\sigma_k \\ \rho_{21}\sigma_2\sigma_1 & \sigma_2^2 & \cdots & \rho_{2k}\sigma_2\sigma_k \\ \cdots & & & \\ \rho_{k1}\sigma_k\sigma_1 & \rho_{k2}\sigma_k\sigma_2 & \cdots & \sigma_k^2 \end{bmatrix} \quad (2)$$

The cumulative distribution function is expressed as

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)[(u-\mu)/\sigma]^2}\, du \quad (3)$$

which is the integral of the density function above [3].

Due to the complexity of the density function, one can see that this is a difficult expression to integrate. For this reason, alternate methods to obtain the values for the cumulative distribution function are necessary.

SIMULATION METHODOLOGY

The simulation methodology employs a simulation routine for obtaining the values for the standard multivariate normal cumulative distribution function. The methodology does not involve a solution or approximation for evaluating the density function described above. The methodology instead performs a simulation of random standard multivariate normal values, based on user input parameters (for their specific standard multivariate normal distribution being evaluated), such as means, standard deviations, and correlation coefficient. From these random standard multivariate normal values, the standard multivariate normal cumulative distribution function values are computed.

The simulation essentially creates random standard normal distribution variables (using traditional methods for generating these random variables), and then relates them using a transformation of the variance-covariance matrix (to be defined later as the **C** matrix). As a result, a standard multivariate normal random variable is created. Then, for a particular desired value of z (that is, the standardized multivariate normal distribution, or z-score value), the simulation program tracks whether this standard multivariate normal random variable is less than or equal to the desired z value. If it is less than or equal to the desired z value (which equates to the definition of the cumulative distribution function), then a counter is increased. This occurs n times. Finally, across all n iterations, the final counter value is obtained, and is then divided by the total number of iterations. The result is the standard multivariate cumulative distribution function value for the specified distribution and z value.

For k >=3, this simulation methodology cannot be used for correlation values that are negative. The cumulative standard multivariate normal distribution values for this range are not discussed in this paper or on the website that is referenced in this paper.

ENDPOINT METHODOLOGY

For the limitations defined in this paper (see "Limitations" section), the simulation methodology described above cannot be used (due to a division by zero constraint in the calculation of the **C** matrix) for k >= 3, ρ = +1. In order to provide cumulative distribution function values for this condition, an alternate to the simulation was developed.

To start, consider the bivariate case (where k = 2, ρ = +1). Using the generic variables A and B to illustrate, it is true that

$$P(A \cap B) = P(A) \cdot P(B|A) \quad \text{for } P(A) > 0 \quad (4)$$

and

$$P(A \cap B) = P(B) \cdot P(A|B) \quad \text{for } P(B) > 0 \quad (5)$$

(where $|$ denotes the conditional probability).

In the case where A=B, then $P(A|B) = P(B|A) = 1$. Then

$$P(A \cap B) = P(A) \quad \text{for } P(A) > 0 \quad (6)$$

and

$$P(A \cap B) = P(B) \quad \text{for } P(B) > 0 \quad (7)$$

It then follows that

$$F(z_1, z_2) = F(z_1) \cdot F(z_2 \mid z_1) = F(z_1) \qquad (8)$$

or

$$F(z_1, z_2) = F(z_2) \cdot F(z_1 \mid z_2) = F(z_2) \qquad (9)$$

so

$$F(z_1, z_2) = F(z_1) = F(z_2) = F(z) \qquad (10)$$

Expanding to the multivariate case (k=3 in this example)

$$F(z_1, z_2, z_3) = F(z_1) \cdot F(z_2, z_3 \mid z_1) = F(z_1) \qquad (11)$$

or

$$F(z_1, z_2, z_3) = F(z_2) \cdot F(z_1, z_3 \mid z_2) = F(z_2) \qquad (12)$$

or

$$F(z_1, z_2, z_3) = F(z_3) \cdot F(z_1, z_2 \mid z_3) = F(z_3) \qquad (13)$$

so

$$F(z_1, z_2, z_3) = F(z_1) = F(z_2) = F(z_3) = F(z) \qquad (14)$$

In general, for $\rho = 1$,

$$F(z_1, z_2, ..., z_k) = F(z) \qquad (15)$$

where $z_1 = z_2 = ... = z_k$.

Therefore, for k >=2, the standard cumulative distribution function value at $\rho = +1$ is equal to the cumulative distribution function value for the standard normal distribution. This must be used in lieu of the tables or Excel functions for k>=2 and $\rho = +1$.

MULTIVARIATE NORMAL VARIABLE GENERATION METHODOLOGY

This section describes how the standard multivariate normal random variables, which are used in the simulation, are generated. First, standard normal random variables are needed, that are used in subsequent calculations. These can be generated individually by first generating twelve random (uniform distribution between 0 and 1) numbers (u_i) and then utilizing the formula

$$z = \sum_{i=1}^{12} u_i - 6 \qquad (16)$$

These z values are used in generating the x values in the equation $x_i = \mu_i + \sum^{i}_{j=1} c_{ij} z_j$ for i = 1 to k, to be described in the following paragraphs.

The multivariate normal distribution has k dimensions. Let μ be the mean vector = $(\mu_1, \mu_2, \mu_3, ..., \mu_k)^T$, and the covariance matrix be Σ. The covariance matrix is a *k* x *k* size matrix, with each (i, j)th entry containing the covariance σ_{ij}. This covariance matrix would appear as

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1k} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2k} \\ \cdots & & & \\ \sigma_{k1} & \sigma_{k2} & \cdots & \sigma_{kk} \end{bmatrix} \qquad (17)$$

Substituting the covariance formulas, $Cov(X_i, X_j) = \sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$, and for the diagonal elements $\sigma_{ii} = \rho_{ii}\sigma_i\sigma_i = \sigma_i^2$ (since $\rho_{ii} = 1$), the matrix would appear as

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1k}\sigma_1\sigma_k \\ \rho_{21}\sigma_2\sigma_1 & \sigma_2^2 & \cdots & \rho_{2k}\sigma_2\sigma_k \\ \cdots & & & \\ \rho_{k1}\sigma_k\sigma_1 & \rho_{k2}\sigma_k\sigma_2 & \cdots & \sigma_k^2 \end{bmatrix} \qquad (18)$$

If $\mathbf{x} = (x_1, x_2, x_3, ..., x_k)^T$ represents any point in k-dimensional real space, then the joint density function can be written as

$$f(\mathbf{x}) = (2\pi)^{-n/2} \left| \Sigma \right|^{-1/2} \exp\{[-(\mathbf{x}-\mu)^T \Sigma^{-1} (\mathbf{x}-\mu)] / 2\} \qquad (19)$$

This multivariate distribution can then be noted as $N_k(\mu, \Sigma)$. If we then define $\mathbf{X} = (X_1, X_2, ..., X_k)^T$ as a distribution that follows the $N_k(\mu, \Sigma)$, then the expected value of X_i would be $E(X_i) = \mu_i$ [6].

Note that Σ is symmetric ($\sigma_{ij} = \sigma_{ji}$) and positive definite [6]. Because of this, it can be factored uniquely as $\Sigma = \mathbf{CC}^T$, where $\mathbf{C}$ is a *k* x *k* matrix and is lower triangular [2]. This matrix would look like

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1k} \\ c_{21} & c_{22} & \cdots & c_{2k} \\ \cdots & & & \\ c_{k1} & c_{k2} & \cdots & c_{kk} \end{bmatrix} \qquad (20)$$

since it is lower triangular

$$\mathbf{C} = \begin{bmatrix} c_{11} & 0 & \dots & 0 \\ c_{21} & c_{22} & \dots & 0 \\ \dots & & & \\ c_{k1} & c_{k2} & \dots & c_{kk} \end{bmatrix} \quad (21)$$

The **C** matrix can then be used to simulate a multivariate normal vector **X**. First, k normal random Z variates are generated. Then, for i = 1 to k, each X_i can be calculated as $X_i = \mu_i + \sum_{j=1}^{i} c_{ij} Z_j$ [6]. Note that this equation can also be written as $X_i = \mu_i + \sum_{j=1}^{k} c_{ij} Z_j$ since the **C** matrix is lower triangular.

The individual **C** matrix elements are calculated according to the following formulas [2]:

$$c_{i1} = \frac{s_{i1}}{(s_{i1})^{1/2}} \quad \text{for } i = 1, \dots, k \qquad A \qquad (22)$$

$$c_{ii} = \sqrt{\left(s_{ii} - \sum_{kk=1}^{i-1} c_{ikk}^{2}\right)} \quad \text{for } i = 2, \dots, k \qquad B \qquad (23)$$

$$c_{ij} = \frac{s_{ij} - \sum_{kk=1}^{j-1} c_{ikk} c_{jkk}}{c_{jj}} \quad \text{for } 1 < j < i <= k \qquad C \qquad (24)$$

$$c_{ij} = 0 \quad \text{for } 0 < i < j <= k \qquad D \qquad (25)$$

Upon viewing these formulas, one can see that many of the **C** matrix elements' calculations are dependent on other **C** matrix elements' values. Because of the interdependencies in the above formulas, the order that the elements are calculated is important.

This can be more easily viewed with a generic matrix (k = 5) example:

$$\mathbf{C}\text{ matrix} = \begin{bmatrix} A & D & D & D & D \\ A & B & D & D & D \\ A & C & B & D & D \\ A & C & C & B & D \\ A & C & C & C & B \end{bmatrix} \quad (26)$$

A represents the elements in the first column (these elements are calculated by the formula "*A*," equation 22, above). The *B* elements are the diagonal elements, with the exception of the first column diagonal element (these elements are calculated by the formula "*B*," equation 23, above). The *C* elements are the elements below the diagonal, with the exception of the first column (these elements are calculated by the formula "*C*," equation 24, above). The *D* elements are the elements above the diagonal, with the exception of the first column (these elements are calculated by the formula "*D*," equation 25, above, which is 0 in the case of the *D* elements).

Adding cell addresses (that is, row, column), it would look like:

$$\mathbf{C}\text{ matrix} = \begin{bmatrix} A_{11} & D_{12} & D_{13} & D_{14} & D_{15} \\ A_{21} & B_{22} & D_{23} & D_{24} & D_{25} \\ A_{31} & C_{32} & B_{33} & D_{34} & D_{35} \\ A_{41} & C_{42} & C_{43} & B_{44} & D_{45} \\ A_{51} & C_{52} & C_{53} & C_{54} & B_{55} \end{bmatrix} \quad (27)$$

The order that the elements would be calculated can now be described. First, the first column is calculated (elements $A_{11}, A_{21}, A_{31}, A_{41}$, and A_{51}). Then, each diagonal element is calculated, followed by the *C* elements in the row from left to right (if there are any *C* elements in the row). So, B_{22} is calculated next. Note that there is no *C* element in row 2. The next diagonal element, B_{33}, is calculated, followed by C_{32}. Similarly, B_{44}, C_{42}, and C_{43} are calculated. The last row is then calculated: B_{55}, C_{52}, C_{53}, and C_{54}. Finally, all *D* elements ($D_{12}, D_{13}, D_{14}, D_{15}, D_{23}, D_{24}, D_{25}, D_{34}, D_{35}$, and D_{45}) are set to zero.

Now that the **C** matrix elements are calculated, they can be used to calculate the multivariate normal vector **X**, using the equation $x_i = \mu_i + \sum_{j=1}^{i} c_{ij} z_j$ for i = 1 to k, as described above. For this study, the values of the means were all set to 0 and the standard deviations set to 1, therefore the multivariate normal vector **X** is actually a standard multivariate normal vector.

CONVERSION OF OBSERVED DISTRIBUTION INTO STANDARD DISTRIBUTION

In order to use the standard multivariate normal distribution cumulative distribution function tables or Excel functions, the user must first convert their observed multivariate normal distribution into the standard multivariate normal distribution. The correlation coefficients must be known, as well as the various means and standard deviation values. The user has information regarding the x_k, σ_k, μ_k and the various ρ values of his data, but needs to determine the corresponding z_k values in order to use the tables or Excel functions to obtain the CDF value.

The website (discussed in the next section) provides spreadsheets that can be used to easily convert the observed distribution into the standard multivariate normal distribution. This section, however, describes the theory behind these spreadsheet tools.

The equation $x_i = \mu_i + \sum_{j=1}^{i} c_{ij} z_j$ for i = 1 to k was discussed in the multivariate normal variable generation section above. This shows the relationship between the x_k values and the z_k values, as related by the elements of the **C** matrix. This equation can therefore be used to convert x_k value observations into their corresponding standard multivariate normal values, z_k.

In matrix form, the relationship of the x_k values, **C** matrix, z_k values and the distribution means μ_k can be written as:

$$[X] = [C]\,[Z] + [\mu] \qquad (28)$$

where [X] is a k x 1 matrix, [C] is a k x k matrix, [Z] is a k x 1 matrix, and [μ] is a k x 1 matrix. Note that this is very similar to the familiar equation used to convert normal distribution observations (that is, k = 1), x, into standard normal distribution variables, z. This equation for the normal distribution is $x = \sigma z + \mu$. In the multivariate scenario, the σ is essentially replaced by the **C** matrix, which is logical since the **C** matrix is a derivation of the variance-covariance matrix, å (that is, å = $\mathbf{CC}^T$, as described in the previous section).

To show the expansion of the matrix form into single equations, the bivariate case will be shown. The matrix forms can then be shown as:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \qquad (29)$$

The individual equations can be written as

$$x_1 = c_{11} z_1 + c_{12} z_2 + \mu_1 \qquad (30)$$
$$x_2 = c_{21} z_1 + c_{22} z_2 + \mu_2 \qquad (31)$$

These are the same equations that would result if one used the equation $x_i = \mu_i + \sum_{j=1}^{k} c_{ij} z_j$ where i = 1 to k.

To solve for z (when x, **C** and μ are known), the equations must be rewritten as follows

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \qquad (32)$$

then

$$\begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \qquad (33)$$

then

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}^{-1} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \qquad (34)$$

If we define [A] to be the inverse of [C], then $[A] = [C]^{-1}$, and then

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix} \qquad (35)$$

In equation form

$$z_1 = a_{11}(x_1 - \mu_1) + a_{12}(x_2 - \mu_2) \qquad (36)$$
$$z_2 = a_{21}(x_1 - \mu_1) + a_{22}(x_2 - \mu_2) \qquad (37)$$

z_1 and z_2 can now be solved for, and the tables and Excel functions can be used to get the CDF value.

In order to use the procedure above to obtain the z values for the original x observed values, the user needs to compute the **C** matrix. The inputs needed to calculate the **C** matrix are first, the number of distributions k, and second, the various elements of the variance-covariance matrix. In order to calculate the variance-covariance matrix, one needs all of the ρ values and all of the various σ values. The ρ values need to be common if k >= 3 if the CDF tables or Excel functions mentioned in this paper are to be used (further described in the Limitations section). In general, however, the **C** matrix can be calculated for any ρ combinations. Note that in the bivariate case, there is only one ρ, so the issue of common ρ values does not apply.

To summarize, the process of translating various x_k multivariate normal values into their corresponding z_k standard multivariate normal values is:

i. Obtain σ_{ij} and ρ_{ij} values (where i and j are from 1 to k), and the μ_k values, from the observed data (k normal distributions)
ii. Use the σ_{ij} and ρ_{ij} values to calculate the variance-covariance matrix (å)
iii. Calculate the **C** matrix
iv. Calculate the inverse of the **C** matrix, **A**
v. Using the x and μ_k values, and the **A** matrix, use the [Z] = [A] [x - μ] process to calculate the z values which correspond to the x values
vi. Use the z values to obtain the CDF values from the tables or Excel Functions discussed in this paper

WEBSITE - CUMULATIVE DISTRIBUTION FUNCTION VALUES

The tables discussed in this paper are provided on the following website: www.stuart.iit.edu. There is one table presented for each value of k from 2 through 15. The user selects the appropriate table for their observed value of k, then selects the appropriate standard z value, and the appropriate correlation coefficient. The cumulative distribution value for the standard multivariate normal distribution can then be read from the table.

The tables are presented for

k=2: -1 <= ρ <= 1, -4.0 <= z <= +4.0
3 >= k >=15: 0 <= ρ <= 1, -4.0 <= z <= +4.0

The Microsoft Excel functions are also provided on the www.stuart.iit.edu website. The website can be accessed and the Excel functions can be used on a spreadsheet on this website. Instructions for use are provided on the website. To use the Excel function #1 (for common z_k values), the user simply needs to enter the k value, the correlation coefficient, the number of simulations they wish to run, and the z value. The standard multivariate normal cumulative distribution function value is then returned. To use Excel Function #2 (for non-common z_k values), the user enters the same information, except they would enter multiple z values.

The process described in the Conversion section in this paper is also available on a spreadsheet on the www.stuart.iit.edu website. This spreadsheet provides a very simple conversion tool for the user to convert their observed multivariate normal distribution into the standard multivariate normal distribution. The variance-covariance and **C** matrices are computed, and all the necessary information is given so that the Excel functions or tables can readily be used.

The aforementioned multivariate normal tables, Excel functions and the conversion spreadsheets should be available on the www.stuart.iit.edu website in April 2002 [8]. More detailed information on the theoretical background of the paper can be found in Carol Lindee's doctoral thesis, "The Multivariate Standard Normal Distribution - Values for the Cumulative Distribution Function," from the Illinois Institute of Technology, Stuart Graduate School of Business, which will be published in the summer of 2002 [7]. This thesis also contains the aforementioned tables for the cumulative standard multivariate normal distribution.

LIMITATIONS

The tables and Excel functions are to be used only for the standard multivariate normal distribution. Conversion of the observed multivariate normal distribution into the standard multivariate normal is necessary before these tools can be used.

The tables on the www.stuart.iit.edu website are for k values from two through fifteen. The correlation values between all of the multivariate variables must be equal (for example, for k = 3, $\rho_{12} = \rho_{13} = \rho_{23}$). Additionally, the z value must be common for each distribution, (for example, for k = 3, $z_1 = z_2 = z_3$), in order to use the tables.

The Excel functions also require that the correlation coefficients be equal. Excel Function #1 requires that the z value be common for all distributions. Excel Function #2 can accommodate non-common z values. The Excel functions have no limit on the value of k. There is also no limitation on the range of the z value(s). The suggested number of simulations (n) when these Excel functions are used is 400,000.

The Excel functions have the following limitations on the common correlation values:

k=2: -1 <= ρ <= 1
k >=3: 0 <= ρ < 1

For k>=3, ρ = 1, the tables need to be used, or the endpoint formula (described in the "Endpoint Methodology" section of this paper) can be evaluated directly to obtain the cumulative distribution function value.

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